Algebraic and combinatorial aspects of face numbers and Stanley-Reisner rings

Lecture 1:

① Polytopes and basic properties

② The upper bound conjecture (UBC)

③ Proof of the upper bound theorem (UBT) for simple / simplicial polytopes

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① Polytopes and basic properties

We start with some basic definitions:

→ We work in $\mathbb{R}^d = \{(x_1, \dots, x_d) : x_i \in \mathbb{R}\}$ endowed with the standard topology and the inner produkt.

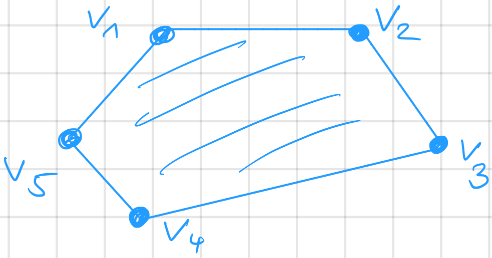
Our protagonists for today will be polytopes.

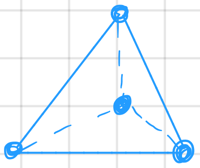
Definition:

A polytope P is the convex hull of finitely many points, i.e.,

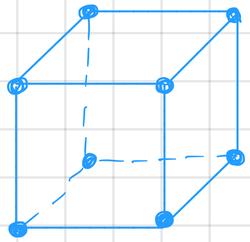
$$P = \operatorname{conv}(\underset{\mathbb{R}^d}{\underset{\cap}{v_1}}, \dots, \underset{\mathbb{R}^d}{\underset{\cap}{v_r}}) = \left\{ \sum_{i=1}^{r} \lambda_i v_i : \lambda_i \geq 0, \sum_{i=1}^{r} \lambda_i = 1 \right\}.$$

Examples

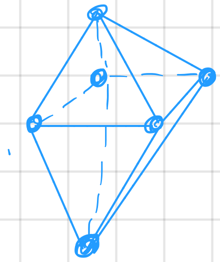




3-simplex



3-cube



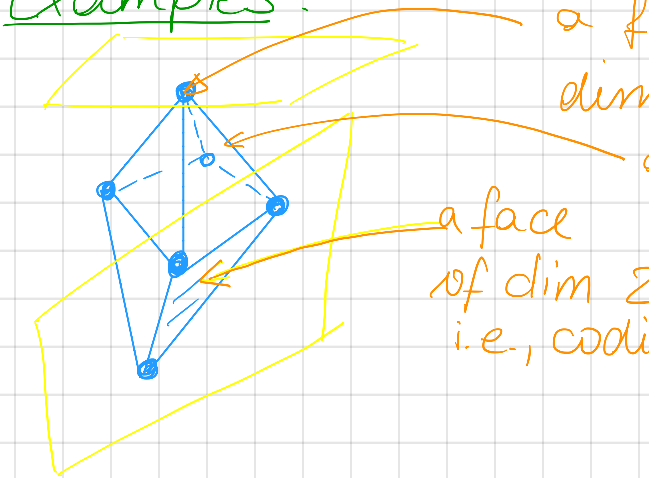
octahedron (= 3-dim'l cross-) polytope

From the pictures we see that every polytope has faces (vertices, edges, ...). Let's make this formal:

Definition:

- A supporting hyperplane of a polytope $P \subseteq \mathbb{R}^d$ is an affine hyperplane $H = \{x \in \mathbb{R}^d : \langle a, x \rangle = b\}$ such that all points of P lie on the same side of H.
- A face of P is the intersection of P with any supporting hyperplane. (Note that $\emptyset$ is a face)
- The dimension of a face F of P is the dimension of $\underbrace{\text{its affine hull}}_{=\text{ smallest affine subspace containing } F} = \text{„translated linear subspace"}$.

Examples:



a face of dim 0 (= vertex)

a face of dim 1 (= edge)

a face of dim 2, i.e., codimension 1 (= facet)

Note: Also $\emptyset$ and P are regarded as (improper) faces with the convention $\dim(\emptyset) = -1$.

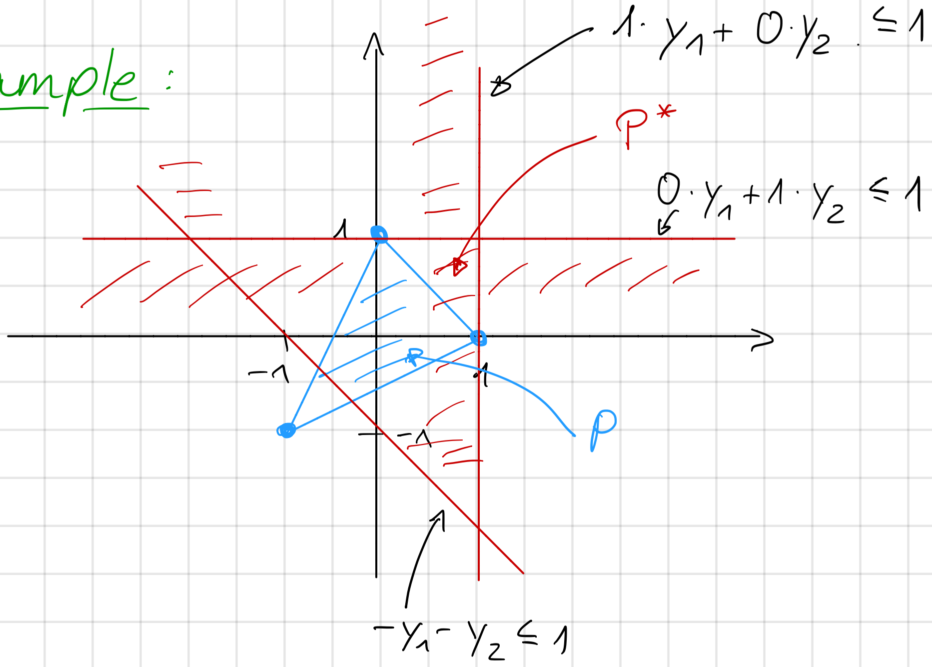
Useful facts

- Every face of a polytope is a polytope.
- The set of faces ordered by inclusion is a graded lattice.
- The set of faces of P ordered by reverse inclusion is the face lattice of a polytope P^* (the (combinatorial) dual or polar) of P.

If we assume that $0 \in \mathrm{Int}(P)$, then we can define P^* via

$$P^* = \{ y \in \mathbb{R}^d \; : \; \langle x, y \rangle \le 1 \text{ for all } x \in P \}$$

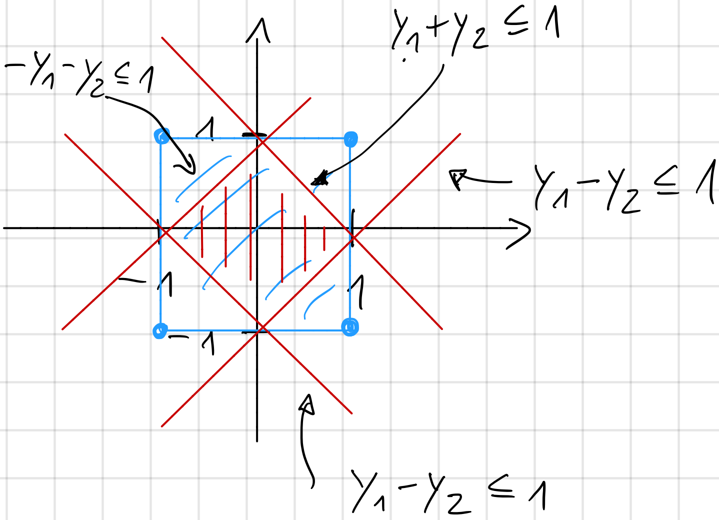
Example:

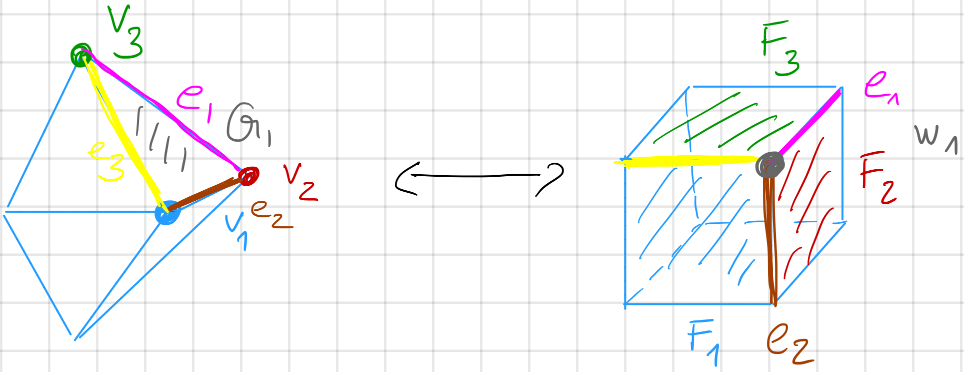


As an exercise you can verify that for $P = [-1,1]^d = \operatorname{conv}(\{-1,1\}^d)$ the d-dimensional cube the dual is given by $P^* = \operatorname{conv}(\{\pm e_i\})$ = the d-dimensional crosspolytope.

↑ i-th unit vector $(0,\dots,0,1,0,\dots,0)$
↑ position i

e.g.





Definition

For a polytope P we call $f(P) = (f_0(P), f_1(P), \dots, \overset{1}{\overset{\shortparallel}{\overbrace{f_{\dim(P)}}}})$ the f-vector of P, where $f_i(P) = \#$ i-dim'l faces of P.

Remark:

We defined, without proving that this is indeed true,

P^* as the (combinatorial) dual of P. Hence,

$$\boxed{f_i(P) = f_{\dim(P)-1-i}(P^*)} \quad \text{for } 0 \le i \le \dim(P)-1 .$$

Example:

We have $f([0,1]^3) = (8,12,6,1)$ and

$f(([0,1]^3)^*) = f(3\text{-crosspolytope}) = (6,12,8,1)$

Simplicial and simple polytopes

- A d-dim'l polytope is simplicial if every face is a $\underbrace{\text{simplex}}_{\text{a polytope whose face lattice is isomorphic to the one of } \operatorname{conv}(e_1,\dots,e_r)}$.

- A d-dim'l polytope is simple if its dual P^* is simplicial.

Remark:

As an exercise one can show that a d-dim'l polytope is simplicial iff one (all) of the following equivalent conditions hold:

(a) every facet of P has d vertices

(b) every proper face of P is a simplex.

(c) every k-face has $k+1$ vertices for $k \le d-1$.

Similarly, a d-dim'l polytope is simple iff one (all) of the following equivalent conditions holds:

(a) Every vertex of P lies in d facets.

(b) Every vertex of P lies in d edges

(c) Every k-face of P lies in d-k facets for $k \geq 0$.

We will use the following easy

Fact:

If P is simple, then so is every face of P.

② The Upper Bound Conjecture

Our protagonists in this part are a family of fascinating polytopes which we now define.

Definition:

(a) The curve $\{ \underbrace{(t, t^2, \ldots, t^d)}_{=: q_d(t)} : t \in \mathbb{R} \}$ is called moment curve in $\mathbb{R}^d$.

(b) Given any n distinct real numbers $t_1 < \ldots < t_n$ the polytope $C(d,n) = \operatorname{conv}(q(t_1), \ldots, q(t_n))$ is called a cyclic polytope.

Here are some properties of $C(d,n)$ that we will not prove in the lecture but that we will consider in the exercises

Properties:

① $\dim C(d,n) = d$ (since any $d+1$ points on the moment curve are seen to be linearly independent using the Vandermonde determinant) and $C(d,n)$ is simplicial.

② $C(d,n)$ is $\lfloor \frac{d}{2} \rfloor$-neighborly, i.e., every collection of $\leq \lfloor \frac{d}{2} \rfloor$ vertices is a face of $C(d,n)$. In particular,

$$f_{k-1}(C_d(n)) = \binom{n}{k} \quad \text{for all } k \leq \lfloor \tfrac{d}{2} \rfloor$$

③ The face lattice of $C(d,n)$ is independent of the chosen points. So, we speak about the cyclic polytope.

④ Gale evenness condition:

A d-tuple $V_d = \{q(t_{i_1}), \dots, q(t_{i_d})\}$ determines a facet $F = \operatorname{conv}(V_d)$ of $C_d(n)$ iff for every 2 points $q(t_i), q(t_j) \in V \setminus V_d$ $(i<j)$

$$|V_d \cap \{q(t_i), q(t_{i+1}), \dots, q(t_j)\}| \text{ is even.}$$

We can now formulate the Upper Bound Conjecture, which is due to Motzkin (1957).

Let P be a d-dim'l (simplicial) polytope with $f_0(P) = n$. Then:

$$\boxed{f_i(P) \leq f_i(C(d,n))}$$

for all $1 \leq i \leq d-1$.

The UBC has been shown for:

- all polytopes (McMullen, 1970)

- all triangulations of $(d-1)$-dim'l spheres (Stanley, 1975)
  (We will prove this tomorrow.)
- all triangulations of odd-dim'l closed manifolds as well as all even-dim'l manifolds of Euler characteristic 2 (Novik, 1998)

Today we will prove the UBC for polytopes. Since by two procedures, called pulling and pushing of vertices, any d-polytope can be converted into a simplicial polytope with the same number of vertices and at least as many j-faces $(1 \leq j \leq d-1)$, we can restrict our attention to simplicial polytopes.

We prove a dual form:

$$C(d,n) \text{ simplicial} \rightsquigarrow C(d,n)^* \text{ simple}$$

$$f_0(C(d,n)) = n \rightsquigarrow f_{d-1}(C(d,n))^* = n$$

Theorem (McMullen, 1970)

Let P be a d-dim'l simple polytope with n facets.

Then: $\boxed{f_i(P) \leq f_i(C(d,n)^*)}$ for all $0 \leq i \leq d-1$.

③ Proof of the UBT

Let $P \subseteq \mathbb{R}^d$ be a d-dim'l simple polytope

Let $\ell : \mathbb{R}^d \to \mathbb{R}$ a linear functional, that is injective on the vertex set of P.

Orient each edge $\{v,w\}$ in the direction of increasing value of ℓ.

Definition:

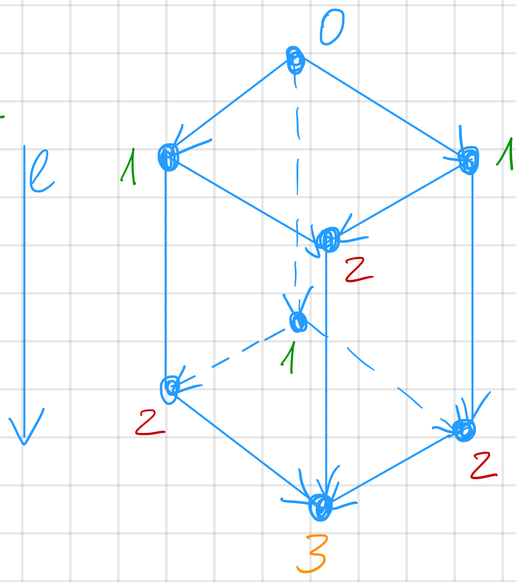
We set $h_k^\ell(P)$ = # of vertices of indegree k

and $h^\ell(P) = (h_0^P, h_1^P, \dots, h_d^P)$

As P is simple each vertex has degree d.

Example:

maximize in direction ℓ



$h^\ell(\text{3-cube}) = (1, 3, 3, 1)$

Theorem:

Let P be a simple d-polytope and ℓ a linear functional as above. Then:

$$\boxed{\sum_{k=0}^{d} f_k(P)\, x^k = \sum_{i=0}^{d} h_i^\ell(P)\,(x+1)^i}$$

In particular, $h_i^\ell(P) = \sum_{k=i}^{d} (-1)^{k-i} \binom{k}{i} f_k(P)$

and $h^\ell(P) =: h(P)$ does not depend on ℓ.

The proof of this theorem uses the following lemma:

Lemma:

Let P be a simple d-polytope and let $\ell : \mathbb{R}^d \to \mathbb{R}$ be a linear functional as before. Let $v \in P$ be a vertex that is a local maximum (i.e., $\ell(v) > \ell(u)$ for all edges $\{u, v\} \in P$), then v is a global maximum. (i.e., $\ell(v) > \ell(u)$ for all vertices $u \in P$).

We will prove both, the lemma and the theorem, in the exercises.

As a consequence of the theorem we obtain.

Corollary:

Let P be a simple d-polytope. Then:

(a) $h_i(P) \geq 0$ for all $0 \leq i \leq d$.

(b) $h_i(P) = h_{d-i}(P)$ for all $0 \leq i \leq d$.

(Dehn - Sommerville equations)

Proof:

(a) is clear from the definition of $h_i(P)$.

(b) $h_i(P) = h_i^\ell(P) = \#(\text{vertices of indegree } i \text{ w.r.t. } \ell)$

$= \#(\text{vertices of indegree } d-i \text{ w.r.t. } -\ell)$

$= h_{d-i}^{(-\ell)}(P) = h_{d-i}(P)$ $\boxtimes$

Observation:

$$f_k(P) = \sum_{i=k}^{d} \underbrace{\binom{i}{k}}_{\geq 0} \underbrace{h_i(P)}_{\geq 0}$$

In particular, bounds for the h-numbers imply bounds for the f-numbers. The UBT hence follows from the following stronger result.

Theorem:

If P is a simple d-polytope with n facets, then

$$(*) \quad \boxed{h_i(P) \leq h_i(C(d,n)^*)} \quad \text{for all } 0 \leq i \leq d.$$

Due to the Dehn-Sommerville equations it suffices to show $(*)$ for $i \geq \lceil \frac{d}{2} \rceil$.

To do so, in the exercises we will compute $h_i(C(d,n)^*)$

Lemma:

$$h_i(C(d,n)^*) = h_{d-i}(C(d,n)^*) = \binom{n-d+i-1}{i} \quad \text{for all } i \leq \lfloor \tfrac{d}{2} \rfloor$$

The previous theorem (and hence the UBT) needs the following two lemmas.

Lemma 1:

Let P be a simple d-polytope and F a facet of P.

Then: $\quad h_i(P) \geq h_{i-1}(F)$ for all $1 \leq i \leq d-1$.

Lemma 2:

Let P be a simple d-polytope. Then for all $0 \le i \le d-1$

$$\sum_{\substack{F \text{ facet} \\ \text{of } P}} h_i(F) = (i+1)h_{i+1}(P) + (d-i)h_i(P).$$

We leave the proofs of Lemmas 1 and 2 as an exercise and instead show how to use them to show $(*)$.

Proof of $(*)$:

We have seen that it suffices to show

$(**)$ $h_{d-i}(P) \le \binom{n-d+i-1}{i}$ for all $i \le \lfloor \frac{d}{2} \rfloor$.

We prove $(**)$ by induction on i.

- $i=0$: $h_d(P) = f_d(P) = 1 = \binom{n-d+0-1}{0}$ ✓
- $i=1$: $h_{d-1}(P) = f_{d-1}(P) - d = \binom{n-d+1-1}{1}$ ✓
- Induction step: $i \longrightarrow i+1$

We have:

$$n \cdot h_{d-i}(P) \overset{\text{Lemma 1}}{\boxed{\ge}} \sum_{F \text{ facet}} \underbrace{h_{d-i-1}(F)}_{\le h_{d-i}(P)}$$

$$\overset{\text{Lemma 2}}{\boxed{=}} (d-i)h_{d-i}(P) + (i+1)h_{d-i-1}(P).$$

$$\Rightarrow (n-d+i)h_{d-i}(P) \ge (i+1)h_{d-i-1}(P)$$

$$\Rightarrow h_{d-i-1}(P) \le \frac{n-d+i}{i+1} h_{d-i}(P)$$

$$\overset{\substack{\text{induction} \\ \text{hypothesis}}}{\boxed{\le}} \frac{n-d+i}{i+1}\binom{n-d+i-1}{i} = \binom{n-d+i}{i+1} \quad \square$$

Lecture 2:

① Simplicial complexes and Stanley-Reisner rings

② The Upper Bound Theorem for Cohen-Macaulay complexes (Stanley's proof) and spheres

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① Simplicial complexes and Stanley-Reisner rings

Our protagonists for today will be simplicial spheres and Cohen-Macaulay complexes.

In order to define these, we first need to introduce some basic notions.

Definition:

- An (abstract) simplicial complex on vertex see V is a collection of subsets of V that is closed under inclusion i.e.,

$$F \in \Delta, G \subseteq F \Rightarrow G \in \Delta.$$

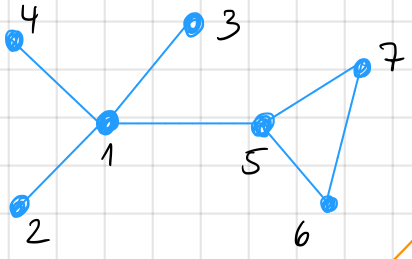
- Elements of Δ are called faces.
- For a face $F \in \Delta$, $\dim F := |F| - 1$ is the dimension of F and $\dim \Delta := \max(\dim F : F \in \Delta)$ is the dimension of Δ.

Examples:

① 0-dimensional simplicial complexes are disjoint unions of n „points".

② 1-dimensional simplicial complexes are just graphs:

$\Delta =$

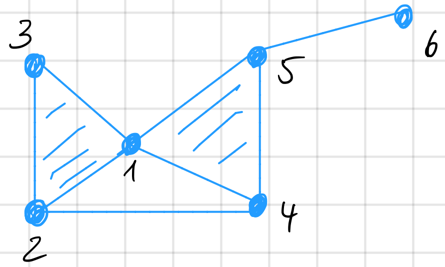


$$= \{\emptyset, \underbrace{\{1\}, \{2\}, \ldots, \{7\},}_{\text{0-dim'l faces = vertices}}$$
$$\underbrace{\{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{5,6\}, \{6,7\}, \{5,7\}\}.}_{\text{1-dim'l faces = edges}}$$

$\emptyset$ is always a face if $\Delta \neq \emptyset$ (of dim -1)

We often omit parantheses and write 123 instead of $\{1,2,3\}$.

③

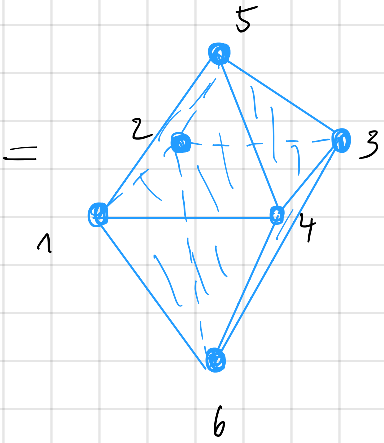


$$\Delta = \{\underbrace{\emptyset,}_{\text{dim}=-1} \underbrace{1, 2, \ldots, 6,}_{\text{dim}=0}$$
$$\underbrace{12, 13, 14, 15, 23, 24, 25, 56,}_{\text{dim}=1}$$
$$\underbrace{123, 145}_{\text{dim}=2}\}$$

dim $\Delta = 2$

④

$\Delta =$



= boundary of the octahedron

$$= \{\emptyset, 1, \ldots, 6, 12, 14, 15, 16, 23, 25, 26,$$
$$34, 35, 36, 45, 46, 125, 126, 145,$$
$$146, 235, 236, 345, 346\}$$

dim $\Delta = 2$

More generally, to any boundary of a simplicial polytope (cf. yesterday's lecture) we can associate a simplicial complex.

As for polytopes, for a $(d-1)$-dimensional simplicial complex Δ we define its *f-vector* $f(\Delta) = (f_{-1}(\Delta), f_0(\Delta), \dots, f_{d-1}(\Delta))$ via

$$f_i(\Delta) = \#\text{ of } i\text{-dim'l faces of } \Delta, \quad -1 \le i \le d-1.$$

In part ②, ③ resp. ④ of the previous example, we have $f(\Delta) = (1,7,7)$, $f(\Delta) = (1,6,8,2)$ resp. $f(\Delta) = (1,6,12,8)$.

In the examples, we have already visualized a simplicial complex geometrically. More generally, this works in the following manner:

Given a simplicial complex Δ on vertex set $V = [n] := \{1,2,\dots,n\}$ we consider $\mathbb{R}^n$ together with its standard basis

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

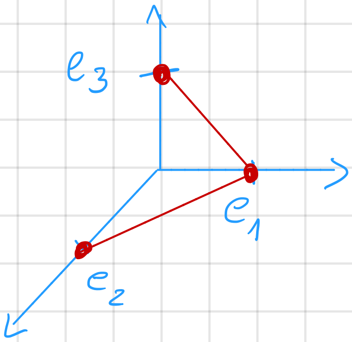
For $F \in \Delta$ define $\underbrace{\|F\| = \operatorname{conv}(e_i : i \in F)}_{\text{is a } (|F|-1)\text{-dim'l simplex}}$.

(in the sense from yesterday)

We set $\|\Delta\| := \bigcup_{F \in \Delta} \|F\|$ and call this the *geometric realization* of Δ.

Example:

$\Delta = \{\emptyset, 1, 2, 3, 12, 13\}$, $\|\Delta\| =$



Remarks:

- $\|\Delta\|$ is a topological space with topology induced from $\mathbb{R}^n$.
- The above construction shows that any simplicial complex

on n vertices can be embedded in $\mathbb{R}^n$. In fact, by choosing n distinct points on the $(2d+1)$-dim'l moment curve any d-dim'l simplicial complex is embeddable (cf. yesterday's lecture) in $\mathbb{R}^{2d+1}$ (but not $\mathbb{R}^{2d}$: e.g., K_5 is only embeddable in $\mathbb{R}^{2\cdot 1+1}$ but not $\mathbb{R}^{2\cdot 1}$).

We can finally define our first protagonist for today:

Definition:

A simplicial sphere is a simplicial complex Δ such that $\|\Delta\|$ is homeomorphic to a sphere.

Example / Comments:

- Any boundary of a simplicial polytope is a simplicial sphere.
- For $d-1 \leq 2$ we have

$$\{(d-1)\text{-dim'l simplicial spheres}\} = \{(d-1)\text{-dim'l } \underbrace{\text{polytopal}}_{\substack{\text{realizable as bounda-}\\ \text{ry of a simplicial polytope}}} \text{ spheres}\}$$

- For $d-1 \geq 3$, most simplicial spheres are not polytopal. ($d-1=3$, Pfeiffle / Ziegler, 2004; $d-1 \geq 4$, Kalai, 1988)

Today's goal: Prove the UBT for simplicial spheres, i.e.,

$$f_i(\Delta) \leq f_i(\underbrace{C(d,n)}_{\substack{\text{the d-dim'l cyclic}\\ \text{polytope on n vertices}}})$$

for any $(d-1)$-dim'l simplicial sphere Δ on n vertices.

For the proof we need to enlarge our toolbox. An extremely useful tool in the study of face numbers is the Stanley-Reisner ring.

Definition:

Let $\mathbb{K}$ be a field and Δ be a simplicial complex on vertex set $[n]$. The Stanley-Reisner ideal I_Δ of Δ is

$$I_\Delta = \langle x_F := \prod_{i \in F} x_i \ : \ F \notin \Delta \rangle \subseteq S := \underbrace{\mathbb{K}[x_1, \dots, x_n]}_{\text{polynomial ring in } n \text{ variables over } \mathbb{K}}.$$

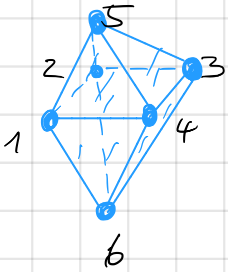
is a squarefree, monomial ideal

$\mathbb{K}[\Delta] := S/I_\Delta$ is called Stanley-Reisner ring or face ring.

Examples:

① If $\Delta = 2^{[d]}$ is a $(d-1)$-dim'l simplex, then $S = \mathbb{K}[x_1, \dots, x_d]$, $I_\Delta = \langle 0 \rangle$ and $\mathbb{K}[\Delta] = S$.

② If $\Delta = \{F \subsetneq [d]\}$ is the boundary of a $(d-1)$-dim'l simplex, then $S = \mathbb{K}[x_1, \dots, x_d]$, $I_\Delta = \langle x_1 \cdots x_d \rangle$ and $\mathbb{K}[\Delta] = S/\langle x_1 \cdots x_d \rangle$.

③ If $\Delta =$ 

= boundary of octahedron, then $S = \mathbb{K}[x_1, \dots, x_6]$, $I_\Delta = \langle x_1x_3, x_2x_4, x_5x_6 \rangle$

Note: Also $\{1,2,3\}$ is not a face but we automatically have $x_1x_2x_3 \in I_\Delta$ since $x_1x_3 \in I_\Delta$.

As generators for I_Δ it suffices to take the ones corresponding to (inclusionwise) minimal non-faces.

Question: Why do we care for $\mathbb{K}[\Delta]$?

Answer: We will see that many combinatorial and topological invariants of Δ are encoded in terms of algebraic invariants of $\mathbb{K}[\Delta]$ and vice versa.

To make this more precise we need some notions from commutative algebra.

Definition:

A *finitely generated, standard, graded $\mathbb{K}$-algebra* is an algebra $R = \bigoplus_{i \in \mathbb{N}} R_i$ such that:

- ▸ $R_0 \cong \mathbb{K}$
- ▸ R_i is a $\mathbb{K}$-vector space
- graded → ▸ $R_i \cdot R_j \subseteq R_{i+j}$
- standard → ▸ R is generated by R_1 (as an algebra)
- finitely generated → ▸ $\dim_{\mathbb{K}} R_1 < \infty$.

It is straigth forward to show that in this case $\dim_{\mathbb{K}} R_i < \infty$ for all i and $R_i R_j = R_{i+j}$.

Example:

$\mathbb{K}[\Delta]$ with the grading induced by the usual degree.

$\mathbb{K}[\Delta]_i = \{ f \in \mathbb{K}[\Delta] \text{ homogeneous of degree } i \}$.

Definition:

▸ For a finitely generated, standard, graded $\mathbb{K}$-algebra $R = \bigoplus_{i \in \mathbb{N}} R_i$ we set $H_R(i) := \dim_{\mathbb{K}} R_i$ for $i \in \mathbb{N}$ and call this the *Hilbert function* of R.

➾ $F_R(t) = \sum_{i \geq 0} H_R(i) \cdot t^i$ is called Hilbert series of R.

In the exercises, we will show the following:

Theorem:

Let Δ be a $(d-1)$-dim'l simplicial complex.

Then,

$$\boxed{F_{\mathbb{K}[\Delta]}(t) = \frac{\sum_{i=0}^{d} f_{i-1}(\Delta) \cdot t^i (1-t)^{d-i}}{(1-t)^d} .}$$

As the numerator is a polynomial in t of degree $\leq t$, we can write it as $\sum_{i=0}^{d} h_i(\Delta) t^i$.

Definition:

$h(\Delta) = (h_0(\Delta), \ldots, h_d(\Delta))$ is called h-vector of Δ.

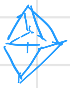
It is a good exercise to show the following explicit formulas:

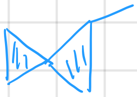
$$\boxed{\begin{aligned} h_i(\Delta) &= \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{i-j} f_{j-1}(\Delta), \quad 0 \leq i \leq d \\ f_{i-1}(\Delta) &= \sum_{j=0}^{i} \binom{d-j}{i-j} h_j(\Delta), \quad 0 \leq i \leq d. \end{aligned}}$$

Examples:

① $h(d\text{-simplex}) = (1, 0, \ldots, 0)$

② $h(\text{boundary of } d\text{-simplex}) = (1, \ldots, 1)$

③ $h($  $) = (1, 3, 3, 1)$

④ $h($  $) = (1, 3, 1, -1)$

Remarks:

- If Δ is the boundary of a simplicial polytope P, we recover the h-vector of the dual P^* as we defined it yesterday, i.e., $h(\Delta) = \underbrace{h}_{\text{yesterday's definition}}(P^*)$.
- While we have $f(\Delta) \geq 0$ (componentwise), $h(\Delta)$ might have negative entries. (cf. Example ④).
- In the exercises you will see a neat way of how to compute the h-vector, known as Stanley's trick.
- As the f-numbers are nonnegative combinations of the h-numbers, in order to show bounds for $f(\Delta)$ it suffices to show bounds for $h(\Delta)$. For the UBT for simplicial spheres Δ, we will hence show $h_i(\Delta) \leq h_i(C(d,n))$.

Indeed, as the Dehn-Sommerville equations ($h_i(\Delta) = h_{d-i}(\Delta)$) hold for simplicial spheres and not only boundaries of simplicial polytopes, it suffices to show the following statement:

Upper Bound Theorem:

Let Δ be a $(d-1)$-dim'l simplicial sphere with $f_0(\Delta) = n$. Then,

$$h_i(\Delta) \leq h_i(C(d,n)) = \binom{n-d+i-1}{i}$$

for $0 \leq i \leq \lfloor \frac{d}{2} \rfloor$.

② The UBT for Cohen-Macaulay complexes

We will derive the UBT for spheres from the following statement.

Theorem:

we will explain this notion in what follows.

Let $\mathbb{K}$ be an infinite field and Δ be a Cohen-Macaulay complex over $\mathbb{K}$ of dimension $d-1$ with n vertices. Then:

$$\boxed{h_i(\Delta) \leq \binom{n-d+i-1}{i} \text{ for } 0 \leq i \leq d.} \quad (\star)$$

We need to review some commutative algebra. In the following $\mathbb{K}$ will always be an infinite field. An important statement is the following:

Noether Normalization Lemma (NNL):

Let A be a finitely generated, standard graded $\mathbb{F}$-algebra. Then there exist $y_1, \dots, y_r \in A_1$ such that

- $y_1, \dots, y_r$ are algebraically independent over $\mathbb{K}$

  $f(y_1, \dots, y_r) \neq 0$ for every polynomial $f \in \mathbb{K}[x_1, \dots, x_r]$.

- there exist homogeneous $\eta_1, \dots, \eta_s$ s.t.

  $$A = \sum_{i=1}^{s} \eta_i \underbrace{\mathbb{K}[y_1, \dots, y_r]}_{B}$$

  (i.e., A is a finitely generated module over $\mathbb{K}$).

Intuitively, a big part of A behaves like a polynomial ring but there are dependencies between different $\eta_i B$.

Definition: In the previous setting, r is called Krull dimension of A, denoted $\dim A$.

There are several other ways to define the Krull dimension. Some are stated in the next theorem.

Theorem:

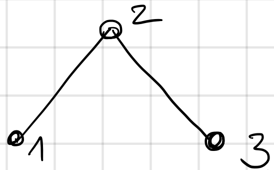
$$\dim A = \max \# \text{ of algebraically independent elements of } A$$
$$= \text{the order to which } t=1 \text{ is a pole of } F_A(t).$$

As an immediate consequence of the second characterization we obtain $\dim \mathbb{K}[\Delta] = \dim \Delta + 1$ if Δ is a simplicial complex.

Definition/Lemma

Elements $y_1, \dots, y_r$ as in the NNL are called linear system of parameters (l.s.o.p.). Equivalently, im $\dim A = r$, then $\dim_{\mathbb{K}} A/\langle y_1, \dots, y_r \rangle < \infty$.

Example:



One can check that x_1, x_2 is not an l.s.o.p. but x_1+x_2, x_2+x_3 is. (for $\mathbb{K}[\Delta] = \mathbb{K}[x_1,x_2,x_3]/\langle x_1 x_3 \rangle$)

- $\mathbb{K}[\Delta]/(x_1, x_2) \cong \mathbb{K}[x_3]$
- $\mathbb{K}[\Delta]/\langle x_1+x_2, x_2+x_3 \rangle \cong \operatorname{span}_{\mathbb{K}}\{1, x_2\}$

as $\mathbb{K}$-vector space

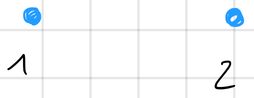
In the exercises, we will see an easy to check criterion if $y_1, \dots, y_d$ is an l.s.o.p. for $\mathbb{K}[\Delta]$. (Kind-Kleinschmidt).

Definition:

There ex. $\eta_1, \dots, \eta_s$ s.t. every $a \in A$ can be uniquely written as $a = \sum \eta_i p_i(y_1, \dots, y_r)$

- A is called Cohen-Macaulay (CM) if A is a free module over $\mathbb{K}[y_1, \dots, y_r]$ for some (every) l.s.o.p. $y_1, \dots, y_r$
- Δ is Cohen-Macaulay over $\mathbb{K}$, if $\mathbb{K}[\Delta]$ is CM.

Example:

① 

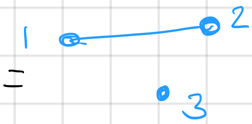
$\mathbb{K}[\Delta] = \mathbb{K}[x_1, x_2] / \langle x_1 x_2 \rangle$, $\dim \mathbb{K}[\Delta] = 1$

- $x_1 + x_2$ is an l.s.o.p. since $\mathbb{K}[\Delta]/\langle x_1+x_2\rangle \cong \operatorname{span}_{\mathbb{K}}\{1, x_1\}$ as $\mathbb{K}$-vector space
- $\mathbb{K}[\Delta]$ is free over $\mathbb{K}[x_1+x_2]$ since

$$x_1^p = \underbrace{x_1}_{=\eta_1}(x_1+x_2)^{p-1} \quad \text{and} \quad x_2^p = \underbrace{1}_{=\eta_2}\cdot(x_1+x_2)^p - \underbrace{x_1}_{=\eta_1}(x_1+x_2)^{p-1}$$

uniquely. We call such an l.s.o.p. regular.

In particular, Δ is CM.

② $\Delta =$ 

Show that Δ is not CM.

$(x_1 + x_3, x_2)$ is an l.s.o.p. but not regular.

Question: Why are CM algebras important?

As an exercise one can show:

Theorem:

Let A be a finitely generated standard, graded algebra $\mathbb{K}$-algebra with l.s.o.p. $y_1, \dots, y_r$. Then:

$$A \text{ is CM} \iff F_A(t) = \frac{F_{A/(y_1,\dots,y_r)}(t)}{(1-t)^r}$$

Note: If A is CM, $F_{A/(y_1,\dots,y_r)}(t)$ is independent of $y_1, \dots, y_r$.

As a first application to simplicial complexes we get:

Corollary:

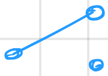
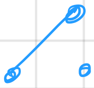
If Δ is CM over some $\mathbb{K}$, then $h_i(\Delta) \geq 0$ for all i.

Proof: Let $\dim \Delta = d-1$ and $y_1, \dots, y_d$ an l.s.o.p. for $\mathbb{K}[\Delta]$.
Set $\mathbb{K}(\Delta) := \mathbb{K}[\Delta]/\langle y_1, \dots, y_d \rangle$. Then

$$\underbrace{F_{\mathbb{K}[\Delta]}(t)}_{\frac{\sum_{i=0}^{d} h_i(\Delta) t^i}{(1-t)^d}} = \frac{F_{\mathbb{K}(\Delta)}(t)}{(1-t)^d} = \frac{\sum_{i \in \mathbb{N}} \dim_{\mathbb{K}} \mathbb{K}(\Delta)_i \, t^i}{(1-t)^d}$$

Hence $h_i(\Delta) = \dim_{\mathbb{K}} \mathbb{K}(\Delta)_i \geq 0$.

∎

Example: $h($  $) = (1,1,-1)$. So  is not CM.

We can now prove (★).

Proof:

Let $y_1, \dots, y_d$ be an l.s.o.p. for $\mathbb{K}[\Delta]$ ← CM of $\dim = d$ with $f_0(\Delta) = n$. Choose $y_{d+1}, \dots, y_n$ such that $y_1, \dots, y_d, y_{d+1}, \dots, y_n$ is a $\mathbb{K}$-basis for $\mathbb{K}[x_1, \dots, x_n]$. The quotient $\mathbb{K}[\Delta]/\langle y_1, \dots, y_d \rangle$ is then generated as an $\mathbb{K}$-algebra by $y_{d+1}, \dots, y_n$.

Hence,

$$\begin{aligned} h_i(\Delta) &= \dim_{\mathbb{K}} \left(\mathbb{K}[\Delta]/\langle y_1, \dots, y_d \rangle \right)_i \\ &\leq \#\text{ monomials of degree } i \text{ in } n-d \text{ variables } (y_{d+1}, \dots, y_n) \\ &= \binom{n-d+i-1}{i} \end{aligned}$$

∎

Note: If Δ is CM and satisfies the Dehn-Sommerville relations, then we get $h_i(\Delta) \leq h_i(C(d,n))$ for all i.

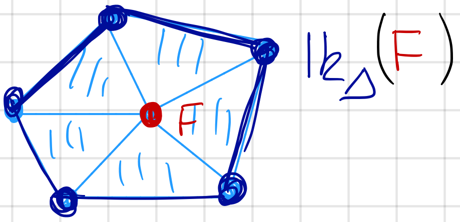
The UBT for spheres finally follows from the following characterization of CM complexes.

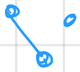
Theorem (Reisner, 1976)

A $(d-1)$-dim'l simplicial complex is CM over $\mathbb{K}$ if and only if

$$\boxed{\tilde{H}_i(\mathrm{lk}_\Delta(F);\mathbb{K}) = 0 \text{ for all } F \in \Delta \text{ and } -1 \leq i < \underbrace{d-|F|-1}_{=\dim \mathrm{lk}_\Delta(F)},}$$

where $\mathrm{lk}_\Delta(F) = \{G \in \Delta \mid G \cap F = \emptyset, G \cup F \in \Delta\}$.



Example: If $\tilde{H}_i(\underbrace{\mathrm{lk}_\Delta(\emptyset)}_{=\Delta};\mathbb{K}) \neq 0$ for $i < \dim \Delta$, then Δ is not CM e.g., 

Corollary:

All simplicial spheres and balls are CM over $\mathbb{K}$.

As simplicial spheres satisfy the Dehn-Sommerville relations we finally get the UBT for spheres (Stanley, 1975)

The UBT for spheres:

Let Δ be a $(d-1)$-dim'l simplicial sphere with n vertices. Then $h_i(\Delta) \leq h_i(C(d,n))$ for all i.
In particular, $f_i(\Delta) \leq f_i(C(d,n))$ for all i.

Lecture 3

① The Kruskal-Katona Theorem: Which integer sequences are f-vectors of simplicial complexes?

② Macaulay's Theorem: Which integer sequences are f-vectors of multicomplexes resp. h-vectors of Cohen-Macaulay complexes?

③ The g-theorem and the Generalized Lower Bound Theorem (GLBT): Which integer sequences are h-vectors of simplicial polytopes? What are lower bounds for such h-vectors?

---

① The Kruskal-Katona Theorem

Our goal is to decide if an integer vector $f=(1,f_0,\ldots,f_{d-1})\in\mathbb{Z}^{d+1}$ is the f-vector of a $(d-1)$-dim'l simplicial complex. For this we need the following lemma.

Lemma:

Given positive integers m and k, there exists a unique expression of m in the following form:

$$\boxed{m=\binom{a_k}{k}+\binom{a_{k-1}}{k-1}+\ldots+\binom{a_s}{s}}$$

k-binomial representation of m

with $a_k > a_{k-1} > \ldots > a_s \geq s \geq 1$.

The proof is a double induction on m and k. We leave it as an exercise.

Example: $m = 21$, $k = 4$

$$21 = \underbrace{\binom{6}{4}}_{=15} + \underbrace{\binom{4}{3}}_{=4} + \underbrace{\binom{2}{2}}_{=1} + \underbrace{\binom{1}{1}}_{=1} .$$

We define

$$\partial_k(m) = \begin{cases} \binom{a_k}{k-1} + \binom{a_{k-1}}{k-2} + \dots + \binom{a_s - 1}{s-1} & \text{if } m \neq 0 \\ 0 & \text{if } m = 0. \end{cases}$$

Example: $\partial_4(21) = \underbrace{\binom{6}{3}}_{=20} + \underbrace{\binom{4}{2}}_{=6} + \underbrace{\binom{2}{1}}_{=2} + \underbrace{\binom{1}{0}}_{=1} = 29$

An answer to our question is provided by the next theorem.

Theorem: (Schützenberger, Kruskal-Katona)
late '50s early '60s

For a vector $f = (1, f_0, f_1, \dots, \overbrace{f_{d-1}}^{\neq 0}) \in \mathbb{Z}^{d+1}$ the following are equivalent:

(a) f is the f-vector of some $(d-1)$-dim'l simplicial complex.

(b) $\partial_{k+1}(f_k) \leq f_{k-1}$ for all $k \geq 1$.

Example: If a simplicial complex has $\overset{f_3}{\overset{\parallel}{21}}$ 3-faces it has at least $\underset{\substack{\wedge \\ f_2}}{29} = \partial_4(21)$ 2-faces, at least $f_1 \geq \partial_3(29)$

$= \binom{6}{3-1} + \binom{4}{2-1} + \binom{3}{1-1} = 22$ edges and at least $f_0 \geq \partial_2(22) = \binom{7}{2-1} + \binom{1}{1-1} = 8$ vertices. Moreover, $(1, 8, 22, 29, 21)$ is the f-vector of a 3-dim'l simplicial complex.

The proof of (b) $\Rightarrow$ (a) is by a direct construction. For this we need several definitions.

- For a family $\mathcal{F}$ of k-subsets of $\mathbb{Z}_{\geq 0}$ we set

$$\underbrace{\partial\mathcal{F} = \{G \subseteq \mathbb{Z}_{\geq 0} \ : \ |G| = k-1 \ , \ G \subseteq F \text{ for some } F \in \mathcal{F}\}}_{\text{the shadow of } \mathcal{F}}$$

Note: Δ is a simpl. complex $\iff \partial\{i\text{-faces}\} \subseteq \{(i-1)\text{-faces}\}$

- revlex-order on k-subsets of $\mathbb{Z}_{\geq 0}$:

$A = \{a_1 < \dots < a_k\} <_{\text{revlex}} \{b_1 < \dots < b_k\} = B \iff \exists i : a_i < b_i$ and $a_j = b_j \ \forall j > i$

- J_k = collection of k-subsets of $\mathbb{Z}_{\geq 0}$ ordered by revlex

$$= \{ \sigma_0 <_{\text{revlex}} <_{\text{revlex}} \sigma_1 <_{\text{revlex}} \dots <_{\text{revlex}} < \sigma_m <_{\text{revlex}} \dots \}$$

Example: $k = 3$, $< \, = \, <_{\text{revlex}}$ (in the following)

$$J_3 = \{012 < 013 < 023 < 123 < 014 < 024 < 124 < 034$$
$$< 134 < 234 < \dots \}$$

For the proof of (b) ⇒ (a) we need two lemmas, whose proofs we defer to the exercises. (Here, Lemma 1 is needed to prove Lemma 2)

Lemma 1:

Let $\sigma_m = \{ \underset{0}{\overset{\vee\!\!|}{a_1}} < \dots < a_k \}$. Then

$$m = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \dots + \binom{a_1}{1} \ , \text{ where}$$

$\binom{a_i}{i} := 0$ if $i > a_i$.

Example: $\{2,3,4\}$ is the $\binom{4}{3} + \binom{3}{2} + \binom{2}{1} = 4+3+2 = 9$th element of J_3.

Lemma 2:

If $\mathcal{F} = \overbrace{\{ \sigma_0 < \sigma_1 < \dots < \sigma_m \}}^{= \text{ initial segment of } J_k} \subseteq J_k$ consists of the first m elements of J_k, then $|\partial\mathcal{F}| = \partial_k(m)$. Moreover, $\partial\mathcal{F}$ is an initial segment of J_{k-1}.

We can now proceed with the proof of (b) $\Rightarrow$ (a):

Given $(1, f_0, \dots, f_{d-1}) \in \mathbb{Z}_{\geq 0}^{d+1}$ with $\partial_{i+1}(f_i) \leq f_{i-1}$ we construct a simplicial complex Δ as follows:

Set Δ_i = first f_{i-1} elements of J_i and $\Delta = \bigcup_{i=1}^{d} \Delta_i \cup \{\emptyset\}$.

As $\partial_{i+1}(f_i) \leq f_{i-1}$, Lemma 2 implies $\partial \Delta_{i+1} \subseteq \Delta_i$ for $i \geq 0$.

Hence, Δ is a simplicial complex. ⊠

Remark:

Simplicial complexes constructed in the previous proof are called compressed. They belong to the more general class of shifted simplicial complexes. Those have a simple combinatorial structure which allows to study alg. + topological properties of those complexes more easily. Operations as algebraic shifting (Kalai, 1983) and combinatorial shifting (Erdős-Ko-Rado) associate a shifted simplicial complex to any simplicial complex while preserving certain properties (f-numbers, Betti numbers, Cohen-Macaulayness, ...)

If $i \in F \in \Delta$ and $1 \leq j < i$, then $F \setminus \{i\} \cup \{j\} \in \Delta$

combinatorial

algebraic

We only sketch the main ideas of the proof of (a) $\Rightarrow$ (b). We follow Frankl's proof (1984) via combinatorial shifting.

- If $\mathcal{A} = \bigcup_{k \geq 0} \mathcal{A}_k$ for $\mathcal{A}_k \subseteq J_k$, we set

$$\partial \mathcal{A} = \bigcup_{k \geq 0} \partial \mathcal{A}_k .$$

Lemma

Let $\mathcal{A}$ be a collection of subsets of $\{0,1,\dots,n\}$. For $0\leq j<n$ and $A\in\mathcal{A}$ set

should be thought of a shift operator replacing j by 0

$$\boxed{S_j(A)} = \begin{cases} (A\setminus\{j\})\cup\{0\} & \text{if } j\in A,\ 0\notin A, (A\setminus\{j\}\cup\{0\})\notin\mathcal{A} \\ A & \text{otherwise}. \end{cases}$$

Let $S_j(\mathcal{A}) = \{S_j(A) : A\in\mathcal{A}\}$.

Then $\partial S_j(\mathcal{A}) \subseteq S_j(\partial\mathcal{A})$.

Note: If $\mathcal{A}$ is a simplicial complex, then so is $S_j(\mathcal{A})$ by Lemma 1 and $f(\mathcal{A}) = f(S_j(\mathcal{A}))$.

The very rough idea of the proof of (a) ⇒ (b) is the following: Let $\Delta = \bigcup_{i=0}^{\dim(\Delta)} \Delta_i \cup \{\emptyset\}$ be a simplicial complex on vertex set $[n]$, where Δ_i = i-dim'l faces

- Apply repeatedly shift operators S_j to Δ $(1\leq j\leq n)$. Since each step increases the number of faces containing 0, after finitely many steps, we get a simplicial complex Δ^* that is stable under S_j, i.e., $S_j(\Delta^*)=\Delta^*$ for all $1\leq i\leq n$ and $f(\Delta^*) = f(\Delta)$.
- Show that

$$f_{\ell-1}(\Delta^*) = |\Delta^*_{\ell-1}| \geq |\partial\Delta^*_\ell| \geq \partial_{\ell+1}|\Delta^*_\ell| = \partial_{\ell+1} f_\ell(\Delta^*).$$

② Macaulay's Theorem: Which integer sequences are f-vectors of multicomplexes resp. h-vectors of Cohen-Macaulay complexes?

Definition:

- A multicomplex on $x_1, \dots, x_n$ is a collection $\mathcal{M}$ of monomials in $x_1, \dots, x_n$ such that:

  (i) $\mu \in \mathcal{M}$ and $\sigma | \mu \Rightarrow \sigma \in \mathcal{M}$

  (ii) $x_i \in \mathcal{M}$ for all $1 \leq i \leq n$.

- For a multicomplex $\mathcal{M}$ we set

$$F_i(\mathcal{M}) = |\{\mu \in \mathcal{M} : \deg(\mu) = i\}$$

and call $F(\mathcal{M}) = (F_0(\mathcal{M}), F_1(\mathcal{M}), \dots)$ the F-vector of $\mathcal{M}$.

Examples:

① Any simplicial complex Δ can be thought of as (squarefree) multicomplex by associating to a face $\{i_1 < \dots < i_k\} \in \Delta$ the monomial $x_{i_1} \cdots x_{i_k}$.
In this case: $f_i(\Delta) = F_{i+1}(\Delta)$.

② $\mathcal{M} = \{1, x, x^2, \dots\}$ is an infinite multicomplex on x_1 with $F(\mathcal{M}) = (1, 1, \dots, 1)$.

③ Let $I \subseteq \mathbb{K}[x_1, \dots, x_n]$ be a monomial ideal and

B_I = set of monomials in $\mathbb{K}[x_1, \dots, x_n]$ not contained in I

Then B_I is a multicomplex. In fact:

I is a monomial ideal $\Longleftrightarrow$ B_I is a multicomplex.

In this case $F_i(B_I) = \dim_{\mathbb{K}} \left(\mathbb{K}[x_1,\dots,x_n]/I\right)_i$

Question: What can we say about F-vectors of multicomplexes?

Before we can give an answer to this question we need one more definition. For

$$m = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \dots + \binom{a_s}{s} \text{ with}$$

$a_k > a_{k-1} > \dots > a_s \geq s \geq 1$, we define

$$m^{\langle k\rangle} = \binom{a_k+1}{k+1} + \binom{a_{k-1}+1}{k} + \dots + \binom{a_s+1}{s+1} \quad \text{and } 0^{\langle k\rangle} = 0.$$

Theorem (Macaulay, 1927)

$F = (F_0, F_1, F_2, \dots) \in \mathbb{Z}_{\geq 0}^{\infty}$ is the F-vector of a multicomplex if and only if $F_0 = 1$ and $0 \leq F_{i+1} \leq F_i^{\langle i\rangle}$ $\forall i \geq 1$.

The proof is very similar to the one by Kruskal-Katona and uses an explicit construction. For $F = (F_0, F_1, \dots)$ one defines $T_i(F)$ = first F_i monomials in $\underbrace{\text{revlex}}$ order of degree i

and $T(F) = \bigcup_{i\geq 0} T_i(F)$.

$x_1^{a_1} \cdots x_n^{a_n} \leq_{\text{revlex}} x_1^{b_1} \cdots x_n^{b_n}$

$\Longleftrightarrow \exists s: a_s < b_s$ and $a_t = b_t$ $\forall t > s$.

One then shows that the following conditions are equivalent:

(i) F is F-vector of a multicomplex

(ii) T_F is a multicomplex

(iii) $F_0 = 1$ and $0 \leq F_{i+1} \leq F_i^{\langle i\rangle}$ $\forall i \geq 1$.

Question: Why are multicomplexes important?

We have seen in Example ③ that F-vectors of multicomplexes B_I are Hilbert functions of quotients of monomial ideals. More generally, in the exercises we will show that given any homogeneous ideal $I \subseteq \mathbb{K}[x_1,\dots,x_n]$, there exists a monomial $\mathbb{K}$-basis B_I of $\mathbb{K}[x_1,\dots,x_n]$ and B_I is a multicomplex. It follows from Example ③ that B_I is the set of monomials not lying in the monomial ideal J and

$$\dim_{\mathbb{K}}(\mathbb{K}[x_1,\dots,x_n]/I)_i = F_i(B_I) = \dim_{\mathbb{K}}(\mathbb{K}[x_1,\dots,x_n]/J)_i.$$

This together with Macaulay's theorem implies:

Theorem:

Let $F = (F_0, F_1, \dots) \in \mathbb{Z}_{\geq 0}^{\infty}$. The following are equivalent:

(a) F is the F-vector of a multicomplex.

(b) $F_0 = 1$, $0 \leq F_{i+1} \leq F_i^{\langle i \rangle}$ $\forall i \geq 1$

(c) F is the Hilbert function of some finitely generated, standard, graded algebra $\mathbb{K}[x_1,\dots,x_n]/J$.

We have seen in the second lecture that if Δ is a $(d-1)$-dim'l Cohen-Macaulay complex with l.s.o.p. $y_1,\dots,y_d$, then

$$h_i(\Delta) = \dim_{\mathbb{K}}\left(\mathbb{K}[\Delta]/\langle y_1,\dots,y_d\rangle\right)_i.$$

So, the previous theorem implies that $h(\Delta)$ satisfies condition (c) if Δ is Cohen-Macaulay. Indeed, Stanley showed the following complete characterization of h-vectors of CM complexes.

Theorem (Stanley, 1975)

Let $h=(h_0,h_1,\dots,h_d)\in\mathbb{Z}^{d+1}$. The following are equivalent:

(a) h is the h-vector of a shellable complex

You have seen this notion in the exercises.

(b) h is the h-vector of a Cohen-Macaulay complex.

Such h is called M-seque

(c) $h_0=1$ and $0\leq h_{i+1}\leq h_i^{\langle i\rangle}$ for all $i\geq 0$.

③ The g-theorem and the Generalized Lower Bound Theorem

Questions: (a) Is there a complete characterization of h-vectors of simplicial polytopes?

(b) What are lower bounds for face numbers of simplicial polytopes?

The answer to (a) and partly (b) is given by the following theorem.

g-theorem (Stanley; Billera-Lee; 1980, conjectured by McMullen 1970)

necessity sufficiency

Let $h=(h_0,h_1,\dots,h_d)\in\mathbb{Z}_{\geq 0}^{d+1}$. The following are equivalent:

(a) There exists a simplicial d-dim'l polytope P such that $h=h(P)$,

(b)
- $h_i=h_{d-i}$ $\forall i$ Dehn-Sommerville relations ← cf. Lecture 1
- $1=h_0\leq h_1\leq\dots\leq h_{\lfloor\frac{d}{2}\rfloor}$ ← Generalized Lower Bound Theorem
- $(\underbrace{h_0}_{=g_0},\underbrace{h_1-h_0}_{=g_1},\dots,\underbrace{h_{\lfloor\frac{d}{2}\rfloor}-h_{\lfloor\frac{d}{2}\rfloor-1}}_{=g_{\lfloor\frac{d}{2}\rfloor}})$ is an M-sequence.

Note: By the Dehn-Sommerville relations $h_0,\dots,h_{\lfloor\frac{d}{2}\rfloor}$ determine $f(\Delta)$ for Δ = boundary of d-dim'l simpl. polytope

Sketch of the proof of the necessity part:

Step 1: Assume that the vertices of P have rational coordinates.
(This can be achieved by slightly perturbing the vertices.)
Let $(p_{i1},\dots,p_{id})$ be the coordinates of vertex i.

Step 2: Set $\theta_i = p_{1i}x_1 + p_{2i}x_2 + \dots + p_{ni}x_n$ for $1 \le i \le d$.
It follows from the Kind–Kleinschmidt criterion we have seen in the exercises that $\theta_1,\dots,\theta_d$ is an l.s.o.p. for $\mathbb{R}[\partial P]$.
and hence $h_i(\partial P) = \dim_{\mathbb{R}} \mathbb{R}[\partial P]/(\theta_1,\dots,\theta_d)$

Note that ∂P is CM.

Step 3:

Fact (Danilov, 1978)

$\mathbb{R}[\partial P]/\langle\theta_1,\dots,\theta_d\rangle$ is isomorphic to the singular cohomology ring of the toric variety X_P corresponding to P.
As X_P is known to satisfy the Hard Lefschetz Theorem it follows that for $\omega = x_1+\dots+x_n$ the following multiplication maps are injective:

$$\times\omega : \left(\mathbb{R}[\partial P]/\langle\theta_1,\dots,\theta_d\rangle\right)_i \longrightarrow \left(\mathbb{R}[\partial P]/\langle\theta_1,\dots,\theta_d\rangle\right)_{i+1}$$

for $0 \le i \le \lfloor \frac{d}{2} \rfloor - 1$. In particular,

$$h_i(\partial P) \le h_{i+1}(\partial P) \qquad \text{for } 0 \le i \le \lfloor \tfrac{d}{2} \rfloor - 1$$

Moreover,
$$\dim_{\mathbb{R}} \left(\mathbb{R}[\partial P]/\langle\theta_1,\dots,\theta_d,\omega\rangle\right)_i = h_i(\partial P) - h_{i-1}(\partial P) = g_i(\partial P),$$

which implies that $(g_0(\partial P),\dots,g_{\lfloor\frac{d}{2}\rfloor}(\partial P))$ is the Hilbert function of $\mathbb{R}[\partial P]/(\langle\theta_1,\dots,\theta_d,\omega\rangle + \overset{\langle x_1,\dots,x_n\rangle}{m}^{\lfloor\frac{d}{2}\rfloor+1})$ and thus an M-sequence. ☒

The sufficiency part of the g-theorem was shown by a direct construction.

Remark:

- The g-theorem has been conjectured to be true for simplicial spheres for a long time. This is known as the g-conjecture.
- In the last 6 months there appeared 3 preprints announcing proofs in general (Adiprasito, 12/2018), for PL-spheres (Karu, 5/2019 and Adiprasito-Steinmeyer, 06/2019)

The answer to question (b) is known as the Generalized Lower Bound Theorem which we now state including the equality case (Murai-Nevo, 2013; conjectured by McMullen/Walkup 1971)

Generalized Lower Bound Theorem

Let P be a d-dim'l simplicial polytope. Then

$$h_0(\partial P) \le h_1(\partial P) \le \dots \le h_{\lfloor \frac{d}{2} \rfloor}(\partial P).$$

Moreover, $h_{i-1}(\partial P) = h_i(\partial P)$ for some $0 \le i \le \lfloor \frac{d}{2} \rfloor$ if and only if P is $(i-1)$-stacked.

i.e., there ex. a triangulation of P without new faces of dimension $\le d-i$

Lecture 4

① Basic properties of balanced simplicial complexes

② Balanced Cohen-Macaulay complexes

③ The balanced generalized lower bound theorem

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① Basic properties of balanced simplicial complexes

We start with the definition of our protagonist for today.

Definition: (this def. goes back to Stanley; he assumed that Δ is pure)

A $(d-1)$-dim'l simplicial complex Δ is called balanced if the graph of Δ is d-colorable, i.e., there exists a map $\kappa: V(\Delta) \to [d]$ such that $\kappa(i) \neq \kappa(j)$ for all $\{i,j\} \in \Delta$.

Note: Since the graph of a $(d-1)$-simplex is a complete graph on d vertices, we cannot color a $(d-1)$-dim'l simplicial complex with less than d colors.

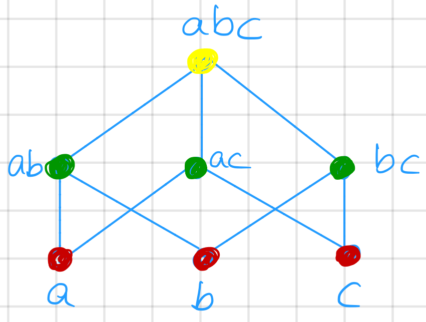
Examples:

faces = chains $p_1 < \ldots < p_k$ with $p_i \in P$

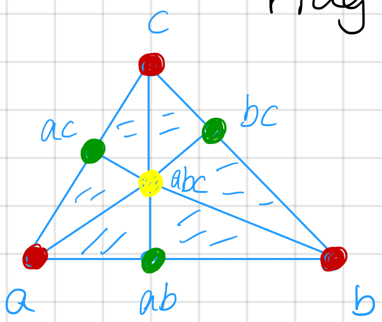
$\updownarrow$

① The order complex $\mathcal{O}(P)$ of a graded poset P of rank d is a $(d-1)$-dim'l balanced simplicial complex, where the coloring is by the rank

$P =$



$\rightsquigarrow \mathcal{O}(P)$

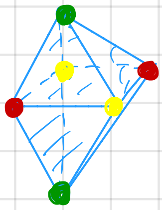


e.g., barycentric subdivision

② Cross-polytopes: $C_d^* = \mathrm{conv}\{\pm e_1, \dots, \pm e_d\}$

A coloring of the boundary complex ∂C_d^* is given by

$\kappa : \{\pm e_1, \dots, \pm e_d\} \longrightarrow [d] : \pm e_i \mapsto i$.

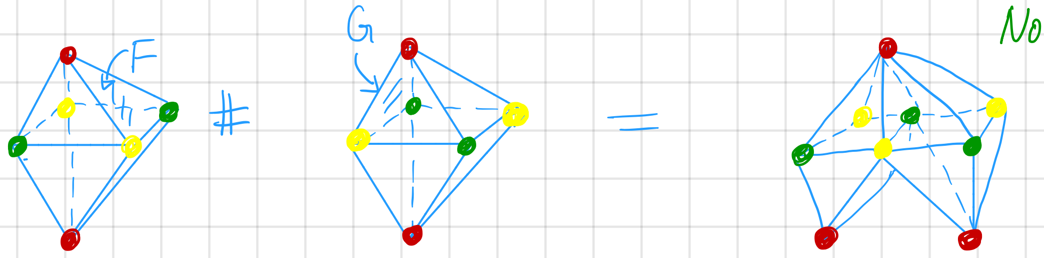


③ Connected sums of balanced complexes:

Δ, Γ $(d-1)$-dim'l balanced simplicial complexes, $F \in \Delta$, $G \in \Delta$ facets with $\varphi : F \to G$ bijection that is color-preserving

The balanced connected sum $\Delta \# \Gamma$ is the balanced simplicial complex obtained by identifying vertices of F on G (and all faces on those vertices) according to φ and removing the fact $F (= G)$.

e.g., stacked cross-polytopal spheres = balanced connected sums of ∂C_d^*.



Note: the face numbers are independent of how we stack but there are different combinatorial types

For balanced simplicial complexes it is common to study the following refinement of the f- and the h-vector.

Definition:

For a $(d-1)$-dim'l balanced simplicial complex Δ with coloring κ we set $\alpha_S(\Delta) = \#\{F \in \Delta : \kappa(F) = S\}$ for $S \subseteq [d]$

= # of faces colored with S

and call $(\alpha_S(\Delta))_{S \subseteq [d]}$ the flag f-vector of Δ.

Moreover, we set $\beta_S(\Delta) = \sum_{T\subseteq S} (-1)^{|S\setminus T|} \cdot \alpha_T(\Delta)$ for $S\subseteq[d]$ and $(\beta_S(\Delta))_{S\subseteq[d]}$ is called flag h-vector.

Note: ▸ $\alpha_S(\Delta) = \sum_{T\subseteq S} \beta_T(\Delta)$

▸ $f_{i-1}(\Delta) = \sum_{\substack{S\subseteq[d]\\ \#S=i}} \alpha_S(\Delta)$, so $(\alpha_S(\Delta))_{S\subseteq[d]}$ refines $f(\Delta)$.

The next lemma shows that $(\beta_S(\Delta))_{S\subseteq[d]}$ is a refinement of $h(\Delta)$.

Lemma: $h_i(\Delta) = \sum_{\substack{S\subseteq[d]\\ \#S=i}} \beta_S(\Delta)$

Sketch of the proof:

We fix variables $\lambda_1,\dots,\lambda_d$. For $T\subseteq[d]$ set $\lambda^T = \prod_{i\in T}\lambda_i$.

One can show that

$$\boxed{\sum_{T\subseteq[d]} \alpha_T(\Delta)\cdot\lambda^T(1-\lambda)^{[d]\setminus T} = \sum_{T\subseteq[d]} \beta_T(\Delta)\,\lambda^T}$$

This is a multivariate/flag version of the usual relation btw. f und h.

Setting $\lambda_i = \frac{1}{x}$ for $1\le i\le d$ and multiplying by x^d we get:

$$\underbrace{x^d\cdot\sum_{T\subseteq[d]}\alpha_T(\Delta)\cdot\frac{1}{x^{|T|}}\cdot\left(1-\frac{1}{x}\right)^{d-|T|}}_{=\sum_{T\subseteq[d]}\alpha_T(\Delta)\cdot(x-1)^{d-|T|}} = \underbrace{x^d\cdot\sum_{T\subseteq[d]}\beta_T(\Delta)\cdot\frac{1}{x^{|T|}}}_{=\sum_{i=0}^{d}\Big(\sum_{\substack{T\subseteq[d]\\ \#T=i}}\beta_T(\Delta)\Big)x^{d-i}}$$

$$=\sum_{i=0}^{d}\Big[\sum_{\substack{T\subseteq[d]\\ \#T=i}}\alpha_T(\Delta)\Big](x-1)^{d-i} = \sum_{i=0}^{d} f_{i-1}(\Delta)(x-1)^{d-i}$$

$$= \sum_{i=0}^{d} h_i(\Delta)\,x^{d-i}$$

$$\Rightarrow h_i(\Delta) = \sum_{\substack{T\subseteq[d]\\ \#T=i}}\beta_T(\Delta)$$

$\square$

As an exercise one can show the following topological interpretation of the flag h-vector:

$$\overbrace{\tilde{\chi}}^{= \sum_{i=0}^{|S|} (-1)^{i-1} f_{i-1}(\Delta_S) = \sum_{i=0}^{|S|-1} (-1)^{i} \tilde{\beta}_i(\Delta_S)}$$

reduced Euler characteristic

(▨) $\beta_S(\Delta) = (-1)^{|S|-1} \cdot \tilde{\chi}(\Delta_S)$,

where $\Delta_S = \{F \in \Delta : \underset{\substack{\uparrow \\ \text{coloring map}}}{\kappa}(F) \subseteq S\}$ for $S \subseteq [d]$ is called rank-selected subcomplex of Δ.

Note: Δ_S is balanced!

$\tilde{\beta}_i(\Delta_S)$
$= \dim_{\mathbb{K}} \tilde{H}_i(\Delta_S; \mathbb{K})$

② Balanced Cohen-Macaulay complexes

The next result states that balanced Cohen-Macaulay complexes behave well when taking rank-selections.

Theorem:

Let Δ be a $(d-1)$-dim'l balanced CM complex with coloring κ and let $S \subseteq [d]$. Then Δ_S is CM of dimension $|S|-1$.

This theorem, together with (▨) and Reisner's criterion (lecture 2) imply the following:

Corollary:

Let Δ be a balanced CM complex. Then: $\boxed{h_i(\Delta) = \sum_{\substack{S \subseteq [d] \\ \#S = i}} \tilde{\beta}_{i-1}(\Delta_S)}$

The next theorem provides a combinatorial characterization of flag h-numbers.

Theorem: ("$\Rightarrow$" Stanley, 1979; "$\Leftarrow$" Björner, Frankl, Stanley 1987)

Let $\beta = (\beta_S)_{S \subseteq [d]} \in \mathbb{Z}^{(2^d)}$. The following are equivalent:

(a) There exists a $(d-1)$-dim'l balanced CM complex such that $\beta(\Delta) = \beta$.

(b) There exists a d-colored simplicial complex Λ such that
$\alpha(\Lambda) = \beta$.

Remark:

- The complex Λ in (b) is not necessarily pure and it can happen that $\dim \Lambda < d-1$, e.g., if $\beta_{[d]} = 0$.
- As a consequence, h-vectors of balanced CM complexes are f-vectors of simplicial complexes and hence satisfy the Kruskal-Katona thm. (those conditions are stronger than the ones from Macaulay's thm.). They satisfy even stronger conditions (Frankl-Füredi-Kalai, 1988)

We only sketch the proof of (a) ⇒ (b). It follows from several propositions. The main new idea is to use that the Stanley-Reisner ring of a balanced simplicial complex Δ is endowed with a $\mathbb{Z}^d$-grading given by

$$\deg(x_j) = \underset{\substack{\uparrow \\ \text{coloring}}}{e_{\kappa(j)}} = (0,\dots,0,\underset{\substack{\uparrow \\ \text{position } \kappa(j)}}{1},0,\dots,0) \in \mathbb{Z}^d.$$

Then, I_Δ is a homogeneous ideal with respect to this grading and hence induces a $\mathbb{Z}^d$-grading on $\mathbb{K}[\Delta]$.

We need a refinement of the Hilbert series to a $\mathbb{Z}^d$-grading.

For variables $\lambda_1, \dots, \lambda_d$, $a = (a_1, \dots, a_d) \in \mathbb{N}^d$ and $S \subseteq [d]$ let

$$\lambda^a := \lambda_1^{a_1} \cdot \ldots \cdot \lambda_a^{a_d} \quad \text{and} \quad \lambda^S := \prod_{i \in S} \lambda_i .$$

Definition:

Let R be a $\mathbb{Z}^d$-graded $\mathbb{K}$-algebra. Then

$$H_R(\lambda_1, \dots, \lambda_d) = \sum_{a \in \mathbb{Z}^d_{\geq 0}} (\dim_{\mathbb{K}} R_a) \lambda^a$$

is called $\mathbb{Z}^d$-graded Hilbert series of R.

Similar to the description of the usual Hilbert series of a Stanley-Reisner ring one gets the following result. (exercise)

Proposition 1:

Let Δ be a $(d-1)$-dim'l balanced simplicial complex with coloring κ. Then

$$H_{\mathbb{K}[\Delta]}(\lambda_1, \dots, \lambda_d) = \frac{\sum_{S \subseteq [d]} \beta_S(\Delta) \lambda^S}{(1-\lambda_1) \cdots (1-\lambda_d)}$$

The next proposition guarantees the existence of a particular nice and simple l.s.o.p. for balanced simplicial complexes.

Proposition 2:

Let Δ be a $(d-1)$-dim'l balanced simplicial complex on vertex set $[n]$. Set $\Theta_i = \underset{\text{coloring} \to \kappa(j)=i}{\sum} x_j$ for $1 \leq j \leq d$. Then:

(i) $\Theta_1, \dots, \Theta_d$ is an l.s.o.p. for $\mathbb{K}[\Delta]$.

(ii) For every $1 \leq j \leq n$: $x_j^2 = 0$ in $\mathbb{K}[\Delta]/\langle \Theta_1, \dots, \Theta_d \rangle$.

Proof:

(i) directly follows from the Kind-Kleinschmidt criterion.

(ii) Let $j \in [n]$ with $\kappa(j) = i$. Then

$$\underbrace{x_j \cdot \Theta_i}_{\in \langle \Theta_1, \ldots, \Theta_d \rangle} = x_j \cdot \Big(\sum_{\kappa(\ell)=i} x_\ell\Big) = x_j \cdot \Big(\sum_{\substack{\kappa(\ell)=i \\ \ell \neq j}} x_\ell + x_j\Big) \overset{\uparrow}{=} x_j^2 \in \mathbb{K}[\Delta].$$

$\{\ell, j\} \notin \Delta$ for all ℓ with $\kappa(\ell) = i$.

Hence: $x_j^2 = 0$ in $\mathbb{K}[\Delta] / \langle \Theta_1, \ldots, \Theta_d \rangle$.

$\square$

As for Hilbert series of quotients of Stanley-Reisner rings by an l.s.o.p. for CM complexes, there is a multigraded analog in the balanced setting.

Proposition 3:

Let Δ be a $(d-1)$-dim'l balanced CM complex with coloring κ. Let $\Theta_1, \ldots, \Theta_d$ be the colored l.s.o.p. as in Proposition 2. Then

$$H_{\mathbb{K}[\Delta]/\langle \Theta_1, \ldots, \Theta_d \rangle}(\lambda_1, \ldots, \lambda_d) = \sum_{S \subseteq [d]} \beta_S(\Delta) \lambda^S.$$

We now sketch the proof of (a) ⇒ (b) of the theorem:

Let $\Theta_1, \ldots, \Theta_d$ be the colored l.s.o.p. and let $a \in \mathbb{Z}^d_{\geq 0}$.

Set

$$\Lambda_a = \left\{ \mu : \begin{array}{l} \mu \text{ monomial of degree s.t.} \\ \mu \notin \underbrace{(I_\Delta)_a}_{\uparrow} + \underbrace{\langle \Theta_1, \ldots, \Theta_d \rangle_a}_{\uparrow} + \langle b : \begin{array}{l} \deg(b) = a \\ b <_{\text{revlex}} \mu \end{array} \rangle \end{array} \right\}$$

degree a component

and $\Lambda = \bigcup_{a \in \mathbb{Z}^d_{\geq 0}} \Lambda_a$.

One shows that Λ is a multicomplex and by Proposition 2 even a simplicial complex. Moreover, Λ is d-colored with the coloring inherited from Δ.

Proposition 3 further implies:

$$\alpha_S(\Lambda) = \dim_{\mathbb{K}} \left(\mathbb{K}[\Delta] / (\theta_1, \dots, \theta_d) \right)_{\underset{\uparrow}{e_S}} = \beta_S(\Delta).$$

$$(e_S)_i = \begin{cases} 1 & i \in S \\ 0 & i \notin S \end{cases}$$

☒

③ The balanced generalized lower bound theorem

In the following we consider simplicial polytopes whose boundary complexes are balanced (balanced simplicial polytopes).

If P is such a balanced simplicial polytope, then we have seen that it satisfies the GLBT:

$$h_0(\partial P) \leq h_1(\partial P) \leq \dots \leq h_{\lfloor \frac{d}{2} \rfloor}(\partial P).$$

It is natural to expect that balancedness forces stronger conditions:

inequalities for h_1, h_2 (together with Goff)

$$\underbrace{\text{(Juhnke-Murai}}_{\text{inequalities}}\text{, 2018; }\underbrace{\text{Klee-Novik}}_{\substack{\text{if-part of equality} \\ \text{only if for } i \leq 2}}\text{, 2016; }\underbrace{\text{Adiprasito}}_{\text{only if-part of equality}}\text{, 2017)}$$

Theorem: Let P be a balanced simplicial polytope of dimension d. Then:

$$\frac{h_0(\partial P)}{\binom{d}{0}} \leq \frac{h_1(\partial P)}{\binom{d}{1}} \leq \dots \leq \frac{h_{\lfloor \frac{d}{2} \rfloor}(\partial P)}{\binom{d}{\lfloor \frac{d}{2} \rfloor}}.$$

Moreover, $\frac{h_{i-1}(\partial P)}{\binom{d}{i-1}} = \frac{h_i(\partial P)}{\binom{d}{i}}$ for some $i \leq \frac{d}{2}$ if and only if P has the $\underbrace{\text{balanced } (i-1)\text{-stacked}}$ property.

Roughly: P can be decomposed into d-dim'l cross-polytopes without introducing interior faces of dimension $\leq d-i$

For $i=1$ we get cross-polytopal stacked spheres.